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On the Existence of Crepant Resolutions for Toric Hyperquotient Singularities in Dimension Three

by

Kohei SATO[†]

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Abstract. In this paper, we show the existence of toric crepant resolutions for singularities which appear in a quotient of an affine toric terminal 3-fold by a finite group acting toroidally.

Key words: crepant resolution, hyperquotient singularities, toric varieties, dimension three.

1. Introduction

Let (X, x) be a normal $\boldsymbol{Q}$-Gorenstein singularity and $f : Y \to X$ be a resolution with exceptional divisors E_i $(i = 1, 2, \dots, r)$. Then the adjunction formula $K_Y = f^*K_X + \sum_{i=1}^{r} a_i E_i$ holds for rational numbers a_i which are called the *discrepancy* for E_i and denoted by $\mathrm{discr}(E_i)$. (X, x) is called *canonical* (resp. *terminal*) if the inequality $\mathrm{discr}(E_i) \geq 0$ (resp. > 0) holds for all i. If f satisfies $\mathrm{discr}(E_i) = 0$ for all i, f is called a *crepant resolution*.

Existence of crepant resolutions is known to be necessary condition for McKay correspondence to hold. The correspondence was found out by [15] as a strange coincidence of graphs for two-dimensional quotient singularities $(\boldsymbol{C}^2/G, 0)$ where G is a finite subgroup of $GL(2, \boldsymbol{C})$. By [7], the coincidence of graphs has been interpreted as an isomorphism between the G-equivariant K-theory of $\boldsymbol{C}^2$ and the K-theory of the minimal resolution of $\boldsymbol{C}^2/G$. Another interpretation in the language of derived categories is in [13]. McKay correspondence was generalized to three-dimensional Gorenstein quotient singularities by using resolutions which are crepant but not minimal. By [10][14][19], it was proved that any three-dimensional Gorenstein quotient singularity $\boldsymbol{C}^3/G$ admits a crepant resolution. It was shown that there exists a correspondence between exceptional divisors of a crepant resolution of $\boldsymbol{C}^3/G$ and conjugacy classes of G in [8]. For three-dimensional Gorenstein quotient singularities, a construction of crepant resolutions by a moduli space named Hilbert scheme of G-orbits and the correspondence by using derived categories was shown by [3]. In the case that the dimension is higher than three, the correspondence does not hold in general because crepant resolutions do not always exist. Nevertheless, for some special

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[†]A member of the Research Fellow of the Japan Society for the Promotion of Science.

cases, sufficient conditions for the existence of a crepant resolution was shown by [4][20] and other papers.

Main objects in the past studies of McKay correspondence were the quotient singularities and we only considered the existence problem of a crepant resolution for those singularities. It would be natural to discuss whether similar properties expand to quotient spaces of terminal 3-folds because a minimal model has terminal singularities in the case that the dimension is equal to or greater than three.

In this paper, we show the existence of toric crepant resolutions for Gorenstein singularities which are given as quotients of affine toric terminal 3-folds X by finite groups G acting *toroidally* (see Definition 2.1) and compute the Euler number of the crepant resolutions. Through this paper, we assume that all varieties are over $\boldsymbol{C}$, and we sometimes denote a singularity (X, x) by X for simplicity because we consider a toric model of the singularity. It is known that any affine toric terminal 3-fold X is isomorphic to either of the two: (i) the quotient singularity of type $\frac{1}{r}(a, -a, 1)$ where a and r are coprime, (ii) the hypersurface singularity $\operatorname{Spec}(\boldsymbol{C}[x, y, z, w]/(xz - yw))$. See Theorem 2.2 and Theorem 2.3. If X is a quotient singularity of type $\frac{1}{r}(a, -a, 1)$ and X/G is a Gorenstein singularity, then there exists a crepant resolution for X/G. This is because the existence problem for X/G can be reduced to the existence problem for $\boldsymbol{C}^3/G'$ where G' is a small finite subgroup of $SL(3, \boldsymbol{C})$ and it is known that all three-dimensional Gorenstein quotient singularities admit a crepant resolution. For details, see Section 3. In the case of the hypersurface $\operatorname{Spec}(\boldsymbol{C}[x, y, z, w]/(xz - yw))$ which we call *the conifold* in this paper, we assume that the quotient X/G has a Gorenstein singularity. In Section 4, we give a classification of the toroidal group actions on the conifold. Finally, in Section 5, we show that X/G admits toric crepant resolutions and compute the Euler number. The main result of this paper is as follows.

THEOREM 1.1. *Let X be an affine toric terminal 3-fold and G be a finite group acting on X toroidally. Assume $(X/G, 0)$ is a Gorenstein singularity. Then X/G admits a toric crepant resolution $\widetilde{X/G}$. If X is the conifold, then the Euler number of $\widetilde{X/G}$ is $2|G|$ where $|G|$ is the order of G.*

We remark that, in the case that X is the conifold, the Euler number of a crepant resolution $\widetilde{X/G}$ is $2|G|$. This implies that the McKay correspondence on $\widetilde{X/G}$ does not hold. Indeed, every conjugacy class of G corresponds to two toric prime divisors on $\widetilde{X/G}$. However this complexity can be interpret by the *string theoretic Hodge theory* and the *strong McKay correspondence* advocated by V. V. Batyrev and D. I. Dais in [2]. A small resolution $\widehat{X/G}$ is covered by two Gorenstein quotient singularities which are isomorphic to each other. Roughly, the strong McKay correspondence is the affine chart-wise McKay correspondence. In this case, if $\widetilde{X/G}$ is covered by two crepant resolutions of the Gorenstein quotient singularities, then $\widetilde{X/G}$ admits the strong McKay correspondence. For the detail, see Section 6.

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2. Preliminary

First of all, we recall basic knowledge of toric varieties (see also [16]) and define *toroidal group actions* on an affine toric variety.

Any toric variety V contains an algebraic torus $\boldsymbol{T}$ as a Zariski open dense subset, and there is an algebraic group action $\pi_{\boldsymbol{T}}$ on V. In this paper, we call $\pi_{\boldsymbol{T}}$ *a torus action* on V. In the case that V is affine, $\boldsymbol{T}$ and the torus orbits of every point in V by $\pi_{\boldsymbol{T}}$ determine an $\boldsymbol{Z}$-module M which is called *character lattice* and a semi-group $S_V \subset M$ respectively. Let $\{m_i \in M : 1 \leq i \leq s\}$ be a system of minimal generators of S_V, hence $S_V = \sum_{i=1}^{s} \boldsymbol{Z}_{\geq 0} m_i$. We have local coordinates $(\boldsymbol{e}(m_1), \ldots, \boldsymbol{e}(m_s))$ on V, and the action $\pi_{\boldsymbol{T}}$ can be written as follows:

$$\pi_{\boldsymbol{T}}(t, (\boldsymbol{e}(m_1), \ldots, \boldsymbol{e}(m_s))) = (t(m_1)\boldsymbol{e}(m_1), \ldots, t(m_s)\boldsymbol{e}(m_s))$$

where t is an element in $\boldsymbol{T}$ and $\boldsymbol{e}(m_i)$ is the character of $\boldsymbol{T}$ for $m_i \in M$.

On the other hand, let H be a finite abelian group acting on V and π_H be the action. If $(V/H, \bar{v})$ is an s-dimensional quotient singularity, then we may assume that V is a vector space over $\boldsymbol{C}$ and G is a subgroup of $GL(s, \boldsymbol{C})$ by the following theorem. See Theorem 6.4.5 in [9].

LEMMA 2.1. *Let (W, w) be an n-dimensional quotient singularity. Then there exists a finite subgroup H of $GL(n, \boldsymbol{C})$ such that $(W, w) \cong (\boldsymbol{C}^n/H, 0)$ as germs. In particular, quotient singularities are algebraic.*

We introduce one more theorem on isomorphisms of quotient singularities (see also [17]).

THEOREM 2.1 (Prill's isomorphism criterion). *Let $H_1, H_2 \subset GL(s, \boldsymbol{C})$ be two small finite subgroups where s is equal to or greater than 2. Then there exists an analytic isomorphism $(\boldsymbol{C}^s/H_1, 0) \cong (\boldsymbol{C}^s/H_2, 0)$ if and only if H_1 and H_2 are conjugate to each other within $GL(s, \boldsymbol{C})$.*

By using the simultaneous diagonalization of all elements in H and Prill's isomorphism criterion, we may assume that the action π_H of H is given as follows:

$$\pi_H(h, c) := \mathrm{diag}(\alpha_1, \ldots, \alpha_s)\, {}^t(c_1, \ldots, c_s)$$

where $\mathrm{diag}(\alpha_1, \ldots, \alpha_s)$ is a diagonalization of the matrix $h \in H$, α_i $(i = 1, \ldots, s)$ are the eigenvalues of h and $(c_1, \ldots, c_s)$ are local coordinates on $\boldsymbol{C}^s$. We note that $\boldsymbol{C}^s$ contains the algebraic torus $(\boldsymbol{C}^*)^s$ and the semigroup $S_{\boldsymbol{C}^s}$ can be written as $S_{\boldsymbol{C}^s} = \sum_{i=1}^{s} \boldsymbol{Z}_{\geq 0} \check{e}_i$ where $\{\check{e}_i : i = 1, \ldots, s\}$ is a $\boldsymbol{Z}$-basis of M. Therefore we can recognize H as a finite subgroup of $\boldsymbol{T}$, and we have the formula:

$$\pi_{\boldsymbol{T}}(t, \pi_H(h, c)) = \pi_H(h, \pi_{\boldsymbol{T}}(t, c)), \quad \forall t \in \boldsymbol{T}, \ \forall h \in H, \ \forall c \in V.$$

This formula says that π_H is equivariant with respect to $\pi_{\boldsymbol{T}}$. In this paper, we consider group actions on affine toric varieties having such equivariance.

DEFINITION 2.1. Let X be an affine toric variety and G be an abelian finite group acting on X. The action π_G is said to be *toroidal* if the action π_G is equivariant with respect to $\pi_{\boldsymbol{T}}$, i.e., π_G satisfies the following formula:

$$\pi_{\boldsymbol{T}}(t, \pi_G(g, x)) = \pi_G(g, \pi_{\boldsymbol{T}}(t, x)), \quad \forall t \in \boldsymbol{T}, \ \forall g \in G, \ \forall x \in X.$$

In addition, we say that G acts on X *toroidally*.

In the following, we assume that all group actions on X are toroidal. The usefulness of toroidal actions is that X/G is also a toric variety. Hence we can describe, explicitly, the toric equivariant geometry of $(X/G, \bar{x})$ with the language of fans.

Let us introduce a classification of affine toric terminal 3-folds. Theorem 2.2 was proved by G. K. White, D. Morrison, G. Stevens, V. Danilov and M. Frumkin and Theorem 2.3 was in [5]. The definition of type $\frac{1}{r}(a, -a, 1)$ follows [18].

THEOREM 2.2. *Let X be an affine toric $\boldsymbol{Q}$-factorial 3-fold. Then X is terminal if and only if X is of type $\frac{1}{r}(a, -a, 1)$ where a is an integer coprime to r. In particular, if X is Gorenstein, then X is smooth.*

THEOREM 2.3. *Let X be an affine toric non-$\boldsymbol{Q}$-factorial 3-fold. Then X has a terminal singularity if and only if $X \cong \mathrm{Spec}(\boldsymbol{C}[x, y, z, w]/(xz - yw))$.*

3. Quotients of affine toric terminal 3-folds of type $\frac{1}{r}(a, -a, 1)$

Let X be an affine toric terminal 3-fold of type $\frac{1}{r}(a, -a, 1)$ and G be a group acting on X toroidally. Then X/G has a toric quotient singularity. We assume that the singularity is Gorenstein. For three-dimensional Gorenstein quotient singularities, the following theorems are known.

THEOREM 3.1 ([10][14][19]). *All three-dimensional Gorenstein quotient singularities admit a crepant resolution.*

By Theorem 3.1, the quotient singularity $(X/G, \bar{x})$ admits a crepant resolution. The remaining problem is whether there really exist quotient morphisms from X to X/G which has a Gorenstein singularity. In the following, we shall give a way to construct an isolated Gorenstein quotient singularity $(X/G, \bar{x})$.

THEOREM 3.2 ([12]). *Let n be an odd prime number. Let H be a finite subgroup of $GL(n, \boldsymbol{C})$ which is small. Assume that the $\boldsymbol{C}^n/H$ is Gorenstein with an isolated singularity. Then $\boldsymbol{C}^n/H$ has a cyclic quotient singularity.*

This theorem is a generalization of Theorem 23 in [21]. By Theorem 2.1 and Theorem 3.2, the singularity $(X/G, \bar{x}) \cong (\boldsymbol{C}^3/\widetilde{G}, 0)$ is a cyclic quotient singularity where $\widetilde{G}$ is a small finite subgroup of $GL(3, \boldsymbol{C})$. So it is enough to prove that, for any cyclic quotient $\boldsymbol{C}^3/\widetilde{G}$, there exists a quotient morphism from X to $\boldsymbol{C}^3/\widetilde{G}$ by a group acting on X toroidally.

Let G' be the finite subgroup of $GL(3, \boldsymbol{C})$ generated by

$$\begin{pmatrix} \varepsilon_r^a & 0 & 0 \\ 0 & \varepsilon_r^{-a} & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}$$

where ε_r is a primitive r-th root of unity and a, r are positive integers which are coprime. Let ϕ_1 be the quotient morphism from $\boldsymbol{C}^3$ to $\boldsymbol{C}^3/G'$, and let ϕ_2 be the quotient morphism from $\boldsymbol{C}^3$ to a cyclic quotient $\boldsymbol{C}^3/\widetilde{G}$. See [Figure 1]. In the following, we shall construct a quotient morphism ϕ such that $\phi \circ \phi_1 = \phi_2$.

$$\begin{tikzcd}
\boldsymbol{C}^3 \arrow[dd, "\phi_2"'] \arrow[dr, "\phi_1"] & \\
& X \arrow[dl, "\phi"] \\
\boldsymbol{C}^3/\widetilde{G} &
\end{tikzcd}$$

Figure 1

We fix the coordinate ring $R := \boldsymbol{C}[x, y, z]$ of $\boldsymbol{C}^3$ on the top of [Figure 1]. The torus action $\pi_{\boldsymbol{T}}$ on $\boldsymbol{C}^3$ defining the coordinate ring $\boldsymbol{C}[x, y, z]$ is as follows:

$$\pi_{\boldsymbol{T}}((t_1, t_2, t_3), (x, y, z)) = (t_1 x, t_2 y, t_3 z)$$

where (t_1, t_2, t_3) is an element in $(\boldsymbol{C}^*)^3$. So it is clear that the action of G' is toroidal for $\pi_{\boldsymbol{T}}$. The following example gives a construction of the quotient morphism ϕ.

EXAMPLE 3.1. Let $G'' \subset GL(3, \boldsymbol{C})$ be the subgroup generated by the matrices

$$\begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}.$$

In [Figure 2], let ϕ_1 and ϕ_2 be the quotient morphisms of $\boldsymbol{C}^3$ by G' and G'' respectively. Since G' is a normal subgroup of G'', there is a quotient morphism from X to $\operatorname{Spec}(\boldsymbol{C}[x^r, y^r, z^r])$.

We define a small finite subgroup $G''' \subset GL(3, \boldsymbol{C})$ as

$$G''' := \left\langle \begin{pmatrix} \varepsilon_{r'}^{a'} & 0 & 0 \\ 0 & \varepsilon_{r'}^{b'} & 0 \\ 0 & 0 & \varepsilon_{r'}^{c'} \end{pmatrix} \right\rangle$$

where $\varepsilon_{r'}$ is a primitive r'-th root of unity, a', b', c' are elements in $\boldsymbol{Z} \cap [0, r')$ and satisfy $\mathrm{GCD}(a', b', c', r') = 1$. G''' clearly acts on $\operatorname{Spec}(\boldsymbol{C}[\check{x}, \check{y}, \check{z}])$, and we denote the quotient morphism form $\operatorname{Spec}(\boldsymbol{C}[\check{x}, \check{y}, \check{z}])$ to $\operatorname{Spec}(\boldsymbol{C}[\check{x}, \check{y}, \check{z}])/G'''$ by ϕ_3. The image of ϕ_3 is of type $\frac{1}{r'}(a', b', c')$.

$$
\begin{tikzcd}
\boldsymbol{C}^3 \cong \mathrm{Spec}(\boldsymbol{C}[x,y,z]) \arrow[dr, "\phi_1"] \arrow[dd, "\phi_2"'] & \\
 & X \arrow[dl] \\
\boldsymbol{C}^3 \cong \mathrm{Spec}(\boldsymbol{C}[x^r,y^r,z^r]) &
\end{tikzcd}
$$

Figure 2

$$
\begin{tikzcd}
\boldsymbol{C}^3 \cong \mathrm{Spec}(\boldsymbol{C}[\check{x},\check{y},\check{z}]) \arrow[d, "\phi_3"'] \\
\boldsymbol{C}^3/G'''
\end{tikzcd}
$$

Figure 3

We shall consider the composition of [Figure 2] and [Figure 3] by the changing of variables

$$x^r \mapsto \check{x},\ y^r \mapsto \check{y},\ z^r \mapsto \check{z}.$$

See [Figure 4]. Let ϕ_4 be the composition of ϕ_2 and ϕ_3. In the following, we ascertain that there really exists ϕ_5 which is the quotient morphism by a group acting on X toroidally.

$$
\begin{tikzcd}
\boldsymbol{C}^3 \arrow[d, "\phi_2"'] \arrow[dr, "\phi_1"] \arrow[dd, bend right=60, "\phi_4"'] & \\
\boldsymbol{C}^3 \arrow[d, "\phi_3"'] & X \arrow[l] \arrow[dl, "\phi_5"] \\
\boldsymbol{C}^3/G''' &
\end{tikzcd}
$$

Figure 4

We define a subgroup $\widetilde{G}$ of $GL(3,\boldsymbol{C})$ as the direct sum of G'' and G''', i.e., $\widetilde{G}$ is as follows:

$$\widetilde{G} = \left\langle \begin{pmatrix} \varepsilon_{rr'}^{r'} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_{rr'}^{r'} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_{rr'}^{r'} \end{pmatrix}, \begin{pmatrix} \varepsilon_{rr'}^{ra'} & 0 & 0 \\ 0 & \varepsilon_{rr'}^{rb'} & 0 \\ 0 & 0 & \varepsilon_{rr'}^{rc'} \end{pmatrix} \right\rangle$$

where $\varepsilon_{rr'}$ is a primitive rr'-th root of unity. Clearly, $\widetilde{G}$ acts on $\mathrm{Spec}(R)$, and $R^{\widetilde{G}}$ coincides with $\boldsymbol{C}[\check{x},\check{y},\check{z}]^{G'''}$ via the changing of variables. This is why ϕ_4 is the quotient morphism from $\mathrm{Spec}(R)$ by $\widetilde{G}$. Since G' is a normal subgroup of $\widetilde{G}$, there exist a quotient map ϕ_5 and a group G acting on X such that $(\boldsymbol{C}^3/\widetilde{G}, 0) \cong (X/G, \bar{x})$. Moreover

$(X/G, \bar{x})$ is a quotient singularity of type $\frac{1}{r'}(a', b', c')$, and G is isomorphic to $\widetilde{G}/G'$. Let $\{u_1(x, y, z), \ldots, u_s(x, y, z)\}$ be a system of minimal generators of $R^{G'}$, i.e., $R^{G'} = \boldsymbol{C}[u_1(x, y, z), \ldots, u_s(x, y, z)]$. The torus action on X is as follows:

$$\pi_{\boldsymbol{T}/G'}((t_1, t_2, t_3), (u_1, \ldots, u_s)) = (u_1(t_1, t_2, t_3)u_1(x, y, z), \ldots, u_s(t_1, t_2, t_3)u_s(x, y, z))$$

where (t_1, t_2, t_3) is an element in $\boldsymbol{T}$. Similarly, the action of G on X is as follows:

$$\begin{aligned}&\pi_G((\varepsilon_{rr'}^{a_1}, \varepsilon_{rr'}^{a_2}, \varepsilon_{rr'}^{a_3}), (u_1, \ldots, u_s))\\ &\quad = (u_1(\varepsilon_{rr'}^{a_1}, \varepsilon_{rr'}^{a_2}, \varepsilon_{rr'}^{a_3})u_1(x, y, z), \ldots, u_s(\varepsilon_{rr'}^{a_1}, \varepsilon_{rr'}^{a_2}, \varepsilon_{rr'}^{a_3})u_s(x, y, z))\end{aligned}$$

where $\varepsilon_{rr'}^{a_i}$ $(i = 1, 2, 3)$ are diagonal components of an element in $\widetilde{G}$. Therefore π_G is toroidal for $\pi_{\boldsymbol{T}/G'}$.

In summary, we have the following.

PROPOSITION 3.1. *Let X be a quotient singularity of type $\frac{1}{r}(a, -a, 1)$ where r and a are coprime. Then there exists a finite group G acting on X toroidally such that X/G has an cyclic quotient singularity of type $\frac{1}{r'}(a', b', c')$ where* $\mathrm{GCD}(a', b', c', r') = 1$.

Additionally, we can prove that X/G has an isolated Gorenstein singularity if and only if $a' + b' + c' \equiv 0 \pmod{r'}$ and $\mathrm{GCD}(a', r') = \mathrm{GCD}(b', r') = \mathrm{GCD}(c', r') = 1$. See Chapter 3 of [21].

4. Toroidal group actions on the conifold

Let X be the conifold $\mathrm{Spec}(\boldsymbol{C}[x, y, z, w]/(xz-yw))$. In this section, we shall classify groups G acting on X toroidally such that X/G has a Gorenstein singularity via a finite fan corresponding to X. The conifold X is a toric variety, and there is a torus action $\pi_{\boldsymbol{T}}$ on X by the algebraic torus $\boldsymbol{T} = (\boldsymbol{C}^*)^3$. The character lattice M is $\boldsymbol{Z}^3$ and S_X is a semi-group $\sum_{i=1}^4 \boldsymbol{Z}_{\geq 0} m_i \subset M$ where m_i $(i = 1, 2, 3, 4)$ are elements in M satisfying $m_1 + m_3 = m_2 + m_4$. In this paper, we choose the canonical $\boldsymbol{Z}$-basis $\check{e_1}, \check{e_2}, \check{e_3}$ of M as m_1, m_2, m_3. Then the semigroup ring $\boldsymbol{C}[S_X]$ is $\boldsymbol{C}[\check{x}, \check{y}, \check{z}, \frac{\check{x}\check{z}}{\check{y}}] \cong \boldsymbol{C}[x, y, z, w]/(xz - yw)$. Let N be the dual $\boldsymbol{Z}$-module of M and $\{e_1, e_2, e_3\}$ be the canonical basis of N. Let Δ be the finite fan which consists of all the faces of the rational strongly convex polyhedral cone $\sigma := \boldsymbol{R}_{\geq 0}e_1 + \boldsymbol{R}_{\geq 0}e_3 + \boldsymbol{R}_{\geq 0}(e_1 + e_2) + \boldsymbol{R}_{\geq 0}(e_2 + e_3)$ in $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. In this case, the finite fan corresponding to X is (N, Δ). We first introduce a lemma on group actions on hypersurfaces (see also Lemma 7.3.8 in [9]).

LEMMA 4.1. *Let $(V, 0) \subset (\boldsymbol{C}^{n+1}, 0)$ be an n-dimensional hypersurface singularity, H be a finite group acting on V and (W, w) be the quotient of $(V, 0)$ by H. Then there exists a small finite subgroup H' of $GL(n+1, \boldsymbol{C})$ such that H' acts on V and $(V, 0)/H' \cong (W, w)$.*

By Lemma 4.1, since the quotient X/G matters, we may assume that the group G is a small finite subgroups of $GL(4, \boldsymbol{C})$. We denote the action of G on X by π_G. The torus action on X is given as $(x, y, z, w) \mapsto (t_1 x, t_2 y, t_3 z, \frac{t_1 t_3}{t_2} w)$ where (x, y, z, w) is a point in

$X \subset \boldsymbol{C}^4$ and $t_1, t_2, t_3 \in \boldsymbol{C}^*$. Therefore, if the action of G is toroidal, then all the elements in G are diagonal 4×4-matrices.

PROPOSITION 4.1. *Let $G \subset GL(4, \boldsymbol{C})$ be a group acting on the conifold X toroidally and g be a generator of G. If the quotient X/G is Gorenstein, then g can be written as*

$$\begin{pmatrix} \varepsilon_r^a & 0 & 0 & 0 \\ 0 & \varepsilon_r^b & 0 & 0 \\ 0 & 0 & \varepsilon_r^{-a} & 0 \\ 0 & 0 & 0 & \varepsilon_r^{-b} \end{pmatrix} \tag{4.1}$$

where ε_r is a primitive r-th root of unity, $a, b \in [0, r) \cap \boldsymbol{Z}$ and $\mathrm{GCD}(a, b, r) = 1$.

Proof. All elements in G are diagonal matrices in $GL(4, \boldsymbol{C})$, and $g \in G$ can be written $\mathrm{diag}(\varepsilon_r^a, \varepsilon_r^b, \varepsilon_r^c, \varepsilon_r^d)$ where r is a positive integer, $a, b, c, d \in [0, r) \cap \boldsymbol{Z}$ and $\mathrm{GCD}(a, b, c, d, r) = 1$.

Since G acts on X, the defining function $xz - yw$ is semi-invariant under π_G. Moreover, since X/G is Gorenstein, the top form $\frac{1}{xz-yw} dx \wedge dy \wedge dz \wedge dw$ is invariant under π_G. By the action of g, $xz - yw$ and $\frac{1}{xz-yw} dx \wedge dy \wedge dz \wedge dw$ are transformed to $\varepsilon^{a+c} xz - \varepsilon^{b+d} yw$ and $\frac{\varepsilon^{a+b+c+d}}{\varepsilon^{a+c} xz - \varepsilon^{b+d} yw} dx \wedge dy \wedge dz \wedge dw$ respectively. Therefore, we have the equations

$$a + c \equiv b + d \quad \textit{and} \quad a + b + c + d \equiv a + c \pmod{r}.$$

In short, the formula

$$a + c \equiv 0 \quad \textit{and} \quad b + d \equiv 0 \pmod{r}$$

holds. Therefore, g can be written as (4.1), and, clearly, the action π_G is toroidal. □

If G is generated by the matrices

$$\begin{pmatrix} \varepsilon_{r_1}^{a_1} & 0 & 0 & 0 \\ 0 & \varepsilon_{r_1}^{b_1} & 0 & 0 \\ 0 & 0 & \varepsilon_{r_1}^{-a_1} & 0 \\ 0 & 0 & 0 & \varepsilon_{r_1}^{-b_1} \end{pmatrix}, \dots, \begin{pmatrix} \varepsilon_{r_k}^{a_k} & 0 & 0 & 0 \\ 0 & \varepsilon_{r_k}^{b_k} & 0 & 0 \\ 0 & 0 & \varepsilon_{r_k}^{-a_k} & 0 \\ 0 & 0 & 0 & \varepsilon_{r_k}^{-b_k} \end{pmatrix}$$

where r_i is a positive integer, ε_{r_i} is a primitive r_i-th root of unity and $a_i, b_i \in [0, r_i) \cap \boldsymbol{Z}$ for $i = 1, \dots, k$, then, by changing r_i into $r := \mathrm{LCM}(r_1, \dots, r_k)$ for all i, these generators of G can be written as follows:

$$\begin{pmatrix} \varepsilon_r^{l_1 a_1} & 0 & 0 & 0 \\ 0 & \varepsilon_r^{l_1 b_1} & 0 & 0 \\ 0 & 0 & \varepsilon_r^{-l_1 a_1} & 0 \\ 0 & 0 & 0 & \varepsilon_r^{-l_1 b_1} \end{pmatrix}, \dots, \begin{pmatrix} \varepsilon_r^{l_k a_k} & 0 & 0 & 0 \\ 0 & \varepsilon_r^{l_k b_k} & 0 & 0 \\ 0 & 0 & \varepsilon_r^{-l_k a_k} & 0 \\ 0 & 0 & 0 & \varepsilon_r^{-l_k b_k} \end{pmatrix}$$

where l_i is $\frac{r}{r_i}$ for all i. In the following, we assume that all elements in G are the ones after the above commonization with respect to r_i. Hence it is enough to classify G for an fixed integer r.

Let us fix an integer r. Let $\mathcal{G}_r$ be the subset of $GL(4, \boldsymbol{C})$ which consists of all diagonal matrices in the form of (4.1) where r is fixed and $a, b \in [0, r) \cap \boldsymbol{Z}$. In this section, we denote the matrix (4.1) by the vector notation $\frac{1}{r}(a, b)$. By the isomorphism $\varphi : \mathcal{G}_r \to (\boldsymbol{Z}/r\boldsymbol{Z})^2$ defined by $\frac{1}{r}(a, b) \mapsto (a, b)$, a finite subgroup G of $\mathcal{G}_r$ corresponds to a finite $(\boldsymbol{Z}/r\boldsymbol{Z})$-submodule. We denote the submodule of $(\boldsymbol{Z}/r\boldsymbol{Z})^2$ corresponding to G by $\bar{G}$. We shall classify subgroups $G \subset \mathcal{G}_r$ into two families: (i) $\{G \subset \mathcal{G}_r : X/G$ is an isolated Gorenstein singularity$\}$, (ii) $\{G \subset \mathcal{G}_r : X/G$ is not so $\}$ via submodules of $(\boldsymbol{Z}/r\boldsymbol{Z})^2$.

Let ψ be the natural surjection $\boldsymbol{Z}^2 \to (\boldsymbol{Z}/r\boldsymbol{Z})^2$. The inverse image $\psi^{-1}(\bar{G})$ is a discrete submodule in $\boldsymbol{R}^2$. Hence, the rank of $\psi^{-1}(\bar{G})$ as a module is at most two, also the cardinality of a system of minimal generators of $\bar{G}$ is at most two. We denote the cardinality of a system of minimal generators of $\bar{G}$ by #SMG($\bar{G}$). For instance, if G is a cyclic group, then #SMG($\bar{G}$) is one. Assume systems of minimal generators $\{(a, b)\}$ (resp. $\{(a, b), (c, d)\}$) of $\bar{G}$ satisfy $\mathrm{GCD}(a, b, r) = 1$ (resp. $\mathrm{GCD}(a, b, c, d, r) = 1$).

PROPOSITION 4.2. *For all submodules $\bar{G}$ of $(\boldsymbol{Z}/r\boldsymbol{Z})^2$ which satisfy* #SMG($\bar{G}$) $=$ 2, *there exists a system of minimal generators which is in the form of either of the following two*:

$$\{(a', b'), (0, d')\}, \tag{4.2}$$

$$\{(a', b'), (c', 0)\} \tag{4.3}$$

where a', b', c', d' are integers in $[0, r) \cap \boldsymbol{Z}$ *and* $\mathrm{GCD}(a', b', c', d', r) = 1$.

Proof. Assume that the set

$$\{(a, b), (c, d)\} \subset (\boldsymbol{Z}/r\boldsymbol{Z})^2$$

is a system of minimal generators of $\bar{G}$ where the integers $a, b, c, d \in [0, r) \cap \boldsymbol{Z}$ and $\mathrm{GCD}(a, b, c, d, r) = 1$. Because #SMG($\bar{G}$) is two, there do not exist non-zero elements $k_1, k_2 \in \boldsymbol{Z}/r\boldsymbol{Z}$ such that $k_1 \cdot (a, b) = (c, d)$ and $k_2 \cdot (c, d) = (a, b)$.

Let α be $\mathrm{GCD}(a, c)$. Then there exist integers $s_1, s_2 \in \boldsymbol{Z} \cap [0, r)$ such that $a = s_1\alpha$ and $c = s_2\alpha$. The integers s_1 and s_2 are coprime. Therefore, there exist integers c_1 and c_2 such that $c_1s_1 + c_2s_2 = 1$, and we have the equation

$$c_1a + c_2c = \alpha .$$

By the above discussion, the formula

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -s_2 & c_1 \\ s_1 & c_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 \\ d' & \beta \end{pmatrix} \pmod r \tag{4.4}$$

holds where $k \equiv s_1 - s_2$, $d' \equiv bc_1 + dc_2$, $\beta \equiv kd' - b'$ and $b' \equiv bs_2 - ds_1$ modulo r. In the formula (4.4), the second matrix, the third one and the fourth one are elementary transformations because those matrices are regular. The integers α, d' and β are elements in $[0, r) \cap \boldsymbol{Z}$. Since GCD is not changed under the elementary transformations of matrices, so we have the equation

$$\mathrm{GCD}(a, b, c, d, r) = \mathrm{GCD}(\alpha, d', \beta, r) = 1 .$$

Clearly, (α, d') and $(0, \beta)$ generate $\bar{G}$. $\square$

We note that if a set $\{(a,0),(0,d) \in \bar{G} : ad \equiv 0 \pmod r\}$ generates $\bar{G}$, then $\#\mathrm{SMG}(\bar{G})$ is one and $\{(a,d) : ad \equiv 0 \pmod r\}$ is a system of minimal generators of $\bar{G}$. Moreover, the converse is also true. We admit interchanging variables x and y while classifying the group actions, and, hence we do not have to distinguish between (4.2) and (4.3). For all submodules $\bar{G}$, there, uniquely, exists a system of minimal generators

$$\mathcal{C}(\bar{G}) := \{(a,b),(0,d)\} \text{ (or } \{(a,b)\}) \subset \bar{G} \tag{4.5}$$

where the integers a,b,d are elements in $[0,r) \cap \boldsymbol{Z}$ which satisfy $\mathrm{GCD}(a,b,d,r) = 1$ (or $\mathrm{GCD}(a,b,r) = 1$) and the conditions: (i) $\mathrm{GCD}(a,r) = a$, (ii) $b = \min\{b' \in [0,r-1] : (a,b') \in \bar{G}\}$, (iii) $\mathrm{GCD}(d,r) = d$. We call the set $\mathcal{C}(\bar{G})$ the *canonical form* of $\bar{G}$ in this paper. In fact, the order of the element (a',b') in (4.2) is $\frac{r}{\mathrm{GCD}(a',r)}$, and, hence, $\mathrm{GCD}(a',r)$ is the minimum of the set $\{a'' \in [1,r-1] : (a'',*) \in \bar{G}\}$. Similarly, $\mathrm{GCD}(d',r)$ is the minimum of the set $\{d'' \in [1,r-1] : (0,d'') \in \bar{G}\}$. Therefore, by the minimality of the integers a,b,d, the uniqueness of the canonical form of $\bar{G}$ follows.

We define submodules C_1 and C_2 of $\bar{G}$ as follows:

$$C_1 := \{(a',0) \in \bar{G}\},\ C_2 := \{(0,d') \in \bar{G}\}\,.$$

We shall explain a relation between the orders of C_1, C_2 and the isolatedness of a singularity X/G by using a finite fan (N',Δ) corresponding to X/G. If the canonical form of $\bar{G}$ is $\mathcal{C}(\bar{G}) = \{(a,b),(0,d)\}$ (resp. $\{(a,b)\}$), then the formula

$$N' = \frac{1}{r}(a,b,-a)\boldsymbol{Z} + \frac{1}{r}(0,d,0)\boldsymbol{Z} + N \quad \left(\text{resp. } \frac{1}{r}(a,b,-a)\boldsymbol{Z} + N\right) \tag{4.6}$$

holds. It is known that there is an isomorphism $G \cong N'/N$ as groups (see [16]). We define $S \subset N_{\boldsymbol{R}}$ as the quadrangle $\{l_1e_1 + l_2(e_1+e_2) + l_3(e_2+e_3) + l_4e_3 : l_1,l_2,l_3,l_4 \in \boldsymbol{R}, \sum_{i=1}^4 l_i = 1\}$. Since $\bar{G}$ is isomorphic to G, there is a surjection ν from N' to $\bar{G}$. We have the restriction of ν on $N' \cap S$ as $\nu|_{N'\cap S}(\frac{1}{r}(s,t,r-s)) \mapsto (s,t)$. We denote the points in N' which are in the inverse image $\nu|_{N'\cap S}^{-1}(\bar{g})$ by $V(\bar{g})$. See [Figure 5] where the largest square is S.

Each cone τ in Δ corresponds to a toric subvariety W of X, and the dimension of τ is equal to the codimension of W. Therefore X has an isolated singularity if and only if the maximal cone σ in Δ is singular and the other cones are not singular. We have the following.

PROPOSITION 4.3. *X/G has an isolated singularity if and only if the canonical form of $\bar{G}$ is as follows*:

$$\mathcal{C}(\bar{G}) = \{(a,b)\} \tag{4.7}$$

where the integers a,b,r are pairwise coprime.

Proof. Assume that G is given by (4.7). Let σ^i be an i-dimensional cone in Δ where $i = 0,1,2,3$. Clearly, σ^i is generated by part of a basis of N' for $i = 0,1$, but the maximal cone $\sigma^3 = \sigma$ is not. There are four two-dimensional cones $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2$ in Δ which are generated by $\{e_2+e_3, e_3\}$, $\{e_1+e_2, e_2+e_3\}$, $\{e_1, e_1+e_2\}$, $\{e_3, e_1\}$ respectively. We denote generators of σ_j^2 ($j = 1,2,3,4$) by $\{g_{1j}, g_{2j}\}$. Since $\mathrm{GCD}(a,r) = \mathrm{GCD}(b,r) = 1$, there is an element $\frac{1}{r}(\alpha_1,\alpha_2,\alpha_3) \in N'$ of which a component α_k is one where $k = 1,2,3$. Let g_{3k}

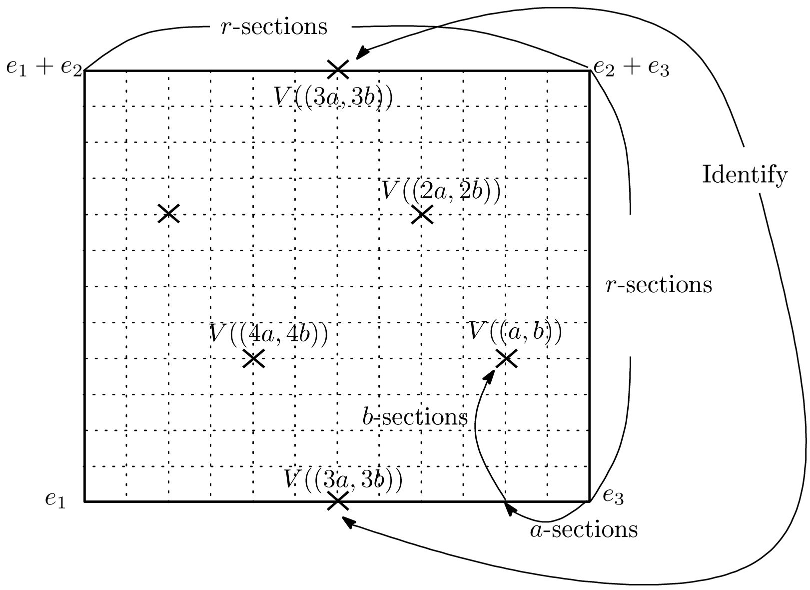

Figure 5

be an element of which k-th component α_k is one. By the formula (4.6), we may assume that the components α_1, α_3 of g_{3k} satisfy the equation $\alpha_1 = -\alpha_3$. By computations, we have the following equations:

$$|\det(g_{1k}g_{2k}g_{3k})| = \frac{1}{r}, \ |\det(g_{14}g_{24}g_{32})| = \frac{1}{r}$$

where $k = 1, 2, 3$. Hence $\{g_{1k}, g_{2k}, g_{3k}\}$ and $\{g_{14}, g_{24}, g_{32}\}$ are bases of N' where $k = 1, 2, 3$, and all two-dimensional cones in Δ are smooth. Therefore X/G has an isolated singularity.

Conversely, we assume that G is given by another canonical form. Let g_1 and g_2 be generators of σ^2. To show that X/G has no isolated singularities, it is enough to prove that there exist a two-dimensional cone $\sigma^2 \in \Delta$ which has an element $h \in N' \cap \sigma^2$ which can not be written as a linear combination of g_1 and g_2 over $\mathbf{Z}$. In fact, assume that there is an element $g_3 \in N'$ such that $\{g_1, g_2, g_3\}$ is a basis of N'. Then h can be written a linear combination of g_1, g_2 and g_3 over $\mathbf{Z}$. Since h, g_1 and g_2 are elements in σ^2, the element g_3 belongs to σ^2. This is a contradiction. Therefore there do not exist elements $g_3 \in N'$ such that $\{g_1, g_2, g_3\}$ is a basis of N'.

If $\#\mathrm{SMG}(\bar{G})$ is two, then N' is given by the equation of the left side of (4.6). The element $\frac{1}{r}(r, d, 0)$ (resp. $\frac{1}{r}(0, d, r)$) belongs to σ_3^2 (resp. σ_1^2), and this element can not be written as a linear combination of g_{13} and g_{23} (resp. g_{11} and g_{21}) over $\mathbf{Z}$. Hence the cones σ_3^2 and σ_1^2 are singular. Hence there are toric subvarieties of X/G which are singular, and the singular locus $\mathrm{Sing}(X/G)$ is not isolated.

Assume that $\#\mathrm{SMG}(\bar{G})$ is one. In this case, by the definition of the canonical form of $\bar{G}$ (see (4.5)), the formula $\mathrm{GCD}(a, b) = 1$ holds. So the remaining are two cases: (i) $\mathrm{GCD}(a, r) \neq 1$, (ii) $\mathrm{GCD}(b, r) \neq 1$. First of all, we consider the case of (i). We denote the integer $\frac{r}{a}$ by l. Since $\mathrm{GCD}(a, b) = 1$, we have an element $l(a, b) = (0, b')$ in $\bar{G}$, and b' is not zero. The lattice point $\frac{1}{r}(0, b', r)$ (resp. $\frac{1}{r}(r, b', 0)$) in the two-dimensional cone σ_1^2 (resp. σ_3^2) can not be written as a linear combination of g_{11} and g_{21} (resp. g_{13} and g_{23}) over $\boldsymbol{Z}$. So the singular locus $\mathrm{Sing}(X/G)$ is not isolated. Finally we treat the case of (ii). Let l' be the integer $\frac{r}{b}$. There is an element $l'(a, b) = (a', 0)$ in $\bar{G}$, and a' is not zero. The lattice point $\frac{1}{r}(a', 0, r - a')$ (resp. $\frac{1}{r}(a', r, r - a')$) in the two-dimensional cone σ_2^2 (resp. σ_4^2) can not be written as a linear combination of g_{12} and g_{22} (resp. g_{14} and g_{24}) over $\boldsymbol{Z}$. $\square$

In other words, X/G has an isolated singularity if and only if both of the orders of the submodules C_1 and C_2 are one. The following is an example that X/G has an isolated singularity $\mathrm{Sing}(X/G)$.

EXAMPLE 4.1. Let X be the conifold and $G \subset GL(4, \boldsymbol{C})$ be a group acting on X toroidally. We assume that $r = 5$ and $\mathcal{C}(\bar{G}) = \{(1, 2)\}$. Let (N', Δ) be the finite fan corresponding to X/G. Then $\mathrm{Sing}(X/G)$ is an isolated singularity. The quadrangle S is as follows.

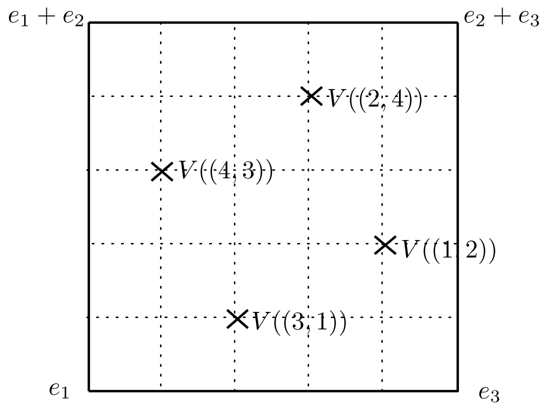

Figure 6

Excepting for the generators of the maximal cone, there are no lattice points on the edge of S. By Proposition 4.3, X/G has an isolated singularity at the origin.

5. The proof of Theorem 1.1

Let X be the conifold and G be a finite group acting on X toroidally. For Note 5.1, see [16] and [18].

NOTE 5.1. Let $f : (\widetilde{\Delta}, N') \to (\Delta, N')$ be a resolution corresponding to a triangulation of S and φ be a support function on $|(\Delta, N')|$. Then we have the formula

$$K_Y = K_X + \sum_{\bar{g}\in\bar{G}} (1 - \varphi(V(\bar{g})))E_{\bar{g}}$$

where X and Y are the toric varieties corresponding to (Δ, N') and $(\widetilde{\Delta}, N')$ respectively and $E_{\bar{g}}$ is the exceptional divisor $\overline{\mathrm{orb}(V(\bar{g}))}$. Moreover, the value of φ is given by $\frac{1}{r}(a+c)$ for $\frac{1}{r}(a,b,c) \in N'$, and $\mathrm{discr}(E_{\bar{g}}) = 0$ for all $\bar{g} \in \bar{G}$.

NOTE 5.2. Let i be an even number, and let P_i be an two-dimensional lattice polygon with i lattice points on its boundary. For any maximal triangulation P_i' of P_i by all lattice points in P_i, the equation

$$2E = 3F + i$$

holds where E (resp. F) is the number of edges (resp. faces) of P_i'.

THEOREM 5.1. *Let G be a finite group acting on the conifold* $\mathrm{Spec}(\boldsymbol{C}[x,y,z,w]/(xz-yw))$ *toroidally. Assume that the quotient* $\mathrm{Spec}(\boldsymbol{C}[x,y,z,w]/(xz-yw))/G$ *has a Gorenstein singularity. Then the quotient admits a toric crepant resolution. Moreover, the Euler number of the crepant resolution is equal to* $2|G|$.

Proof. Isolated case.

Assume that $\mathrm{Spec}(\boldsymbol{C}[x,y,z,w]/(xz-yw))/G$ has an isolated singularity. By Proposition 4.3 and [Figure 5], there are no lattice points on ∂S except for $e_1, e_3, e_1+e_2, e_2+e_3$ where ∂S is the boundary of S. By the relation between $\bar{g} \in \bar{G}$ and $V(\bar{g}) \in N' \cap S$ in the discussion of [Figure 5], the number of the elements in $N' \cap S$ is $|G| + 3$. Let S' be a maximal triangulation of $S \subset N_{\boldsymbol{R}}$ by using all lattice points in $N' \cap S$. By Note 5.2, we have the equation

$$E = \frac{3}{2}F + 2$$

where E (resp. F) is the number of edges (resp. faces) in S'. By combining with the Euler's polyhedral theorem, the formula

$$|G| + 3 - \frac{3}{2}F - 2 + F = 1$$

holds. Therefore, the number of the triangles in S' is $2|G|$.

Let T be a minimal triangle in S' and t_1, t_2, t_3 be the vertices of T. Then the formulas

$$\mathrm{Vol}(T) = |\det(t_1, t_2, t_3)| \geq \frac{1}{r},$$

$$\mathrm{Vol}(S) = |\det(e_1, e_2+e_1, e_2+e_3)| + |\det(e_2+e_3, e_3, e_1)| = 2$$

hold. Therefore, for all minimal triangles T in S', we have the equation

$$\mathrm{Vol}(T) = \frac{1}{r}.$$

This says that every maximal cone in Δ which is generated by the vertices of T is smooth. By Note 5.1 and the above discussion, the triangulation S' determines a toric crepant resolution, and the Euler number of the crepant resolution is equal to $2|G|$.

Non-isolated case.

Suppose that X/G has no isolated singularities. Then the order of C_1 or C_2 is not one, and there are lattice points $V(C_i)$ on ∂S where $i = 1, 2$. The number of lattice points in $N' \cap S$ is $|G| + |C_1| + |C_2| + 1$. Since there are $2(|C_1| + |C_2|)$ edges on ∂S, S can be seen as a lattice polygon $P_{2(|C_1|+|C_2|)}$ (see Note 5.2). By Note 5.2, we have the formula

$$E = \frac{3}{2}F + |C_1| + |C_2|\,.$$

By the Euler's polyhedral theorem, the equation

$$|G| + |C_1| + |C_2| + 1 - \left(\frac{3}{2}F + |C_1| + |C_2|\right) + F = 1$$

holds. Therefore, there are $2|G|$ minimal triangles in S'. By the same reason as the isolated case, this triangulation determines a toric crepant resolution, and its Euler number is $2|G|$. $\square$

By Proposition 3.1, Theorem 3.1 and Theorem 5.1, we have Theorem 1.1.

6. Discussion

Let X and G be as in Theorem 5.1. In this section, we shall consider the strong McKay correspondence for X/G. We note that X/G is a *GV-variety* in Remark 6.4 in [2].

For X, there are two small resolutions $\pi_i : \widehat{X}^i \to X$ $(i = 1, 2)$. Since G acts on X toroidally and the small resolutions are toric, the group action lifts on $\widehat{X}^i$. Moreover, π_i is compatible with the quotient map by G. The small partial resolution of X/G denoted by $\pi' : \widehat{X}/G \to X/G$ $(i = 1, 2)$ is covered by two affine charts X^i_j/G^i_j, $(j = 1, 2)$ on the exceptional set isomorphic to $\boldsymbol{P}^1$ where X^i_j is isomorphic to $\boldsymbol{C}^3$ and G^i_j is isomorphic to a finite subgroup of $SL(3, \boldsymbol{C})$. i.e., X^i_j/G^i_j is a Gorenstein quotient singularity.

$$\widehat{X}^i/G = \bigcup_{j=1}^{2} X^i_j/G^i_j$$

Therefore, for every affine chart X^i_j/G^i_j, there is a crepant resolution $f_{ij} : \widetilde{X^i_j/G^i_j} \to X^i_j/G^i_j$. These two morphisms patch together to give a crepant resolution of the hole X/G.

PROPOSITION 6.1. *Let* X, G *and* X^i_j/G^i_j $(i = 1, 2,\ j = 1, 2)$ *be as above. Then* X^i_1/G^i_1 *is isomorphic to* X^i_2/G^i_2.

Proof. By computing Laurent polynomial rings of affine charts of small resolutions of X, we have the fact that a small resolution of X is covered by two affine charts $\mathrm{Spec}(\boldsymbol{C}[\frac{y}{z}, z, \frac{xz}{y}])$ and $\mathrm{Spec}(\boldsymbol{C}[x, y, \frac{z}{y}])$ or $\mathrm{Spec}(\boldsymbol{C}[\frac{x}{y}, y, z])$ and $\mathrm{Spec}(\boldsymbol{C}[x, \frac{y}{x}, \frac{xz}{y}])$. We set the notations as follows:

$$X^1_1 = \mathrm{Spec}\left(\boldsymbol{C}\left[\frac{y}{z}, z, \frac{xz}{y}\right]\right)\,, \quad X^1_2 = \mathrm{Spec}\left(\boldsymbol{C}\left[x, y, \frac{z}{y}\right]\right)\,,$$

$$X_1^2 = \mathrm{Spec}\left(\boldsymbol{C}\left[\frac{x}{y}, y, z\right]\right), \quad X_2^2 = \mathrm{Spec}\left(\boldsymbol{C}\left[x, \frac{y}{x}, \frac{xz}{y}\right]\right).$$

We assume that the order of G is r and the canoniccal form of G is $\mathcal{C}(\bar{G}) = \{(a, b)\}$. It is easy to check that X_j^1 $(j = 1, 2)$ is the Gorenstein quotient singularities of type $\frac{1}{r}(a, b, -a-b)$ and X_j^2 $(j = 1, 2)$ is the one of type $\frac{1}{r}(a, -b, b-a)$. $\square$

We note that, in general, X_j^1/G_j^1 is not isomophic to X_j^2/G_j^2. If (i) X/G is a non-isolated singularity or (ii) G is cyclic and $|G|$ is even number, then X_j^i/G_j^i is non-isolated singularity. X_j^i/G_j^i is not necessary isolated singularity even if (iii) X/G is an isolated singularity and $|G|$ is odd number, however either X_j^1/G_j^1 or X_j^2/G_j^2 is always an isolated singularity. Figure 7 (resp. Figure 8) is the quadrangle $S := \{l_1 e_1 + l_2(e_1 + e_2) + l_3(e_2 + e_3) + l_4 e_3 : l_1, l_2, l_3, l_4 \in \boldsymbol{R}, \sum_{i=1}^4 l_i = 1\} \subset N'_{\boldsymbol{R}}$ of $\widehat{X}^1/G$ (resp. $\widehat{X}^2/G$) and explains the relation between the way of small resolutions and the isolatedness of X_j^i/G_j^i in the case of (iii).

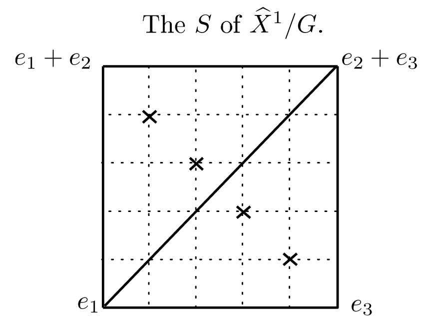

Figure 7

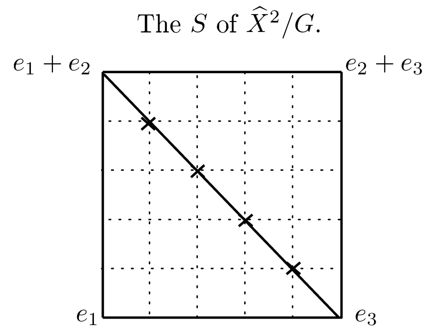

Figure 8

Two triangles in Figure 7 (resp. Figure 8) are corresponding to the quotient singularities of type $\frac{1}{r}(a, b, -a-b)$ (resp. $\frac{1}{r}(a, -b, b-a)$). In the following, we assume that, if G satisfies the condition (iii), then $\widehat{X}/G$ is covered by affine charts with an isolated singularity.

In general, crepant resolution of X/G is not unique. However, by using toric flops, arbitrary crepant resolution of X/G can be reduced to the special ones which are coverd by $\widetilde{X_j^i/G_j^i}$. For every j, the McKay correspondence on $\widetilde{X_j^i/G_j^i}$ stands. It is proved that, if two smooth irreducible projective Calabi-Yau algebraic varieties are birational, then the i-th cohomologies with coefficient $\boldsymbol{C}$ of these varieties are isomorphic. See [1], [11]. Therefore, when we consider the strong McKay correspondence for $\widetilde{X/G}$, it is natural to take the above reduction by toric flops.

Finally, we give an example of the strong Mckay correspondence in the case (iii).

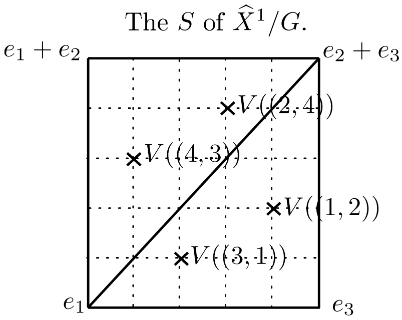

Figure 9

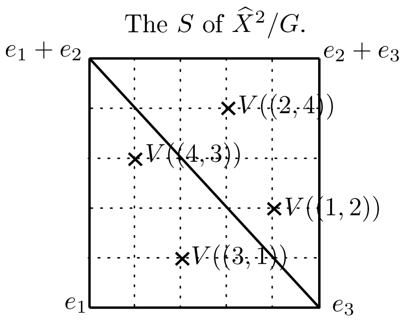

Figure 10

EXAMPLE 6.1. Let X and G be as in Example 4.1. Then the S of a small resolution of X/G is as follows.

In this case, $\widehat{X}^1/G$ (resp. $\widehat{X}^2/G$) has two isolated singularities of type $\frac{1}{5}(1,2,2)$ (resp. $\frac{1}{5}(1,3,1)$).

Let H be an abelian subgroup of $SL(3,\boldsymbol{C})$ generated by the diagonal matrix $\operatorname{diag}(\varepsilon_3, \varepsilon_3^2, \varepsilon_3^2)$. The McKay correspondence for the quotient singularity of type $\frac{1}{5}(1,2,2)$ is as follows:

$$\Gamma_1 = \left\{\frac{1}{5}(1,2,2), \frac{1}{5}(3,1,1)\right\} \leftrightarrow H^2(Y,\boldsymbol{Q})$$

$$\Gamma_1^{(0)} = \left\{\frac{1}{5}(1,2,2), \frac{1}{5}(3,1,1)\right\} \leftrightarrow \Gamma_2 = \left\{\frac{1}{5}(2,4,4), \frac{1}{5}(4,3,3)\right\} \leftrightarrow H^4(Y,\boldsymbol{Q}) \leftrightarrow H_c^2(Y,\boldsymbol{Q})$$

where Y is a crepant resolution of the singularity. Notations follows [8]. The Euler number $e(Y)$ is as follows:

$$e(Y) = h^0(Y,\boldsymbol{Q}) + h^2(Y,\boldsymbol{Q}) + h^4(Y,\boldsymbol{Q}) = 1+2+2 = 5\,.$$

On the other hand, there is a correspondence between H and G.

$$H \supset \Gamma_1 = \left\{\frac{1}{5}(1,2,2), \frac{1}{5}(3,1,1)\right\} \leftrightarrow \left\{\frac{1}{5}(1,2,4,3), \frac{1}{5}(3,1,2,4)\right\} \subset G$$

$$H \supset \Gamma_2 = \left\{\frac{1}{5}(2,4,4), \frac{1}{5}(4,3,3)\right\} \leftrightarrow \left\{\frac{1}{5}(2,4,3,1), \frac{1}{5}(4,3,1,2)\right\} \subset G$$

The figure [Figure 11] is a crepant resolution $\widetilde{X/G}$ of X/G which is covered by crepant resolutions of $\boldsymbol{C}^3/H$.

The S of $\widetilde{X/G}$.

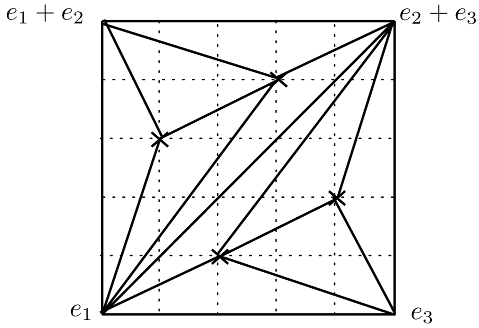

Figure 11

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Department of Mathematics
and Information Sciences
Tokyo Metropolitan University
1–1 Minami-osawa, Hachioji
Tokyo 1920397, Japan
E-mail: sato-kohei@ed.tmu.ac.jp